Building CSPs from semigroups

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The finite case

A rough definition

A **constraint satisfaction problem** (Montanari, 1974) consists of:

1. a finite list of variables V,
2. a domain of possible values A,
3. a set of constraints on those variables $\mathcal{C}$.

Problem: Can we assign values to all the variables so that all the constraints are satisfied?

Example (Graph 3-colouring)

Let G be a finite graph. Each vertex can be coloured either red, green or blue. Problem: can we colour G such that no two adjacent variables have the same colour?

Example (N-Queen)

Place N queens on an $N \times N$ chess board so that no queen can attack any other queen.

Constraint language

Much attention has been paid to the case where the constraints arise from fixed relations on a finite domain.

Definition

Given a finite relational structure $(A; \Gamma)$, we define $\text{CSP}(A; \Gamma)$, or simply $\text{CSP}(\Gamma)$, to be the CSP with:

- **Instance:** $I = (V, A, \mathcal{C})$ in which each constraint is simply a relation from Γ.
- **Question:** Does I have a solution?

Example

Let $\mathcal{G} = (G; E)$ be a finite simple graph. Then an instance of $\text{CSP}(\mathcal{G})$ could be $(x, y), (y, z), (z, x) \in E$.

Example

Let $\mathcal{P} = (P; \leq, U)$ be a finite poset with ternary relation U. Then an instance of $\text{CSP}(\mathcal{P})$ could be $x \leq y, z \leq y, x \leq z, U(x, y, t), U(t, s, z)\ldots.$

Examples

Equivalently, CSP($\mathcal{A}$) is defined as:

Definition

Given a finite relational structure $\mathcal{A} = (A; \Gamma)$, we define CSP($\mathcal{A}$) to be the CSP with:

- **Instance:** A finite structure S of the same relational signature of A.
- **Question:** Does S map homomorphically to A?

Example

Graph 3-colouring can be considered as CSP($A; \Gamma$) where $A = \{R, B, G\}$ has the single binary relation $\Gamma = \{(x, y) \in A : x \neq y\}$. Or, equivalently, considered as CSP(K_3), where K_3 is the complete graph on 3 vertices.

Computational Complexity

Main question: What is the computational complexity of CSP($\mathcal{A}$)?

Definition

1. $\mathbb{P}$: the class of all problems solved in polynomial time. Its members are called **tractable**.
2. $\mathbb{NP}$: the class of problems solvable in nondeterministic polynomial time.
3. $\mathbb{NP}$-hard: the class of problems which at least as hard as the hardest problems in $\mathbb{NP}$.
4. $\mathbb{NP}$-complete: the class of problems which are $\mathbb{NP}$ and $\mathbb{NP}$-hard (the "hardest problems in $\mathbb{NP}$").

We assume $\mathbb{P} \neq \mathbb{NP}$.

Theorem (Ladner, 1975)

If $\mathbb{P} \neq \mathbb{NP}$ then there are problems in $\mathbb{NP} \setminus \mathbb{P}$ that are not $\mathbb{NP}$-complete.

Dichotomy Theorem (!)

As A is finite, CSP(A) is always in $\mathbb{NP}$.

Example

Graph n-colouring is $\mathbb{NP}$-complete if $n > 2$, and tractable otherwise. Equivalently, CSP(K_n) is $\mathbb{NP}$-complete when $n > 2$, and tractable otherwise.

Example (Hell and Nešetři, 90')

Let G be a finite undirected graph. Then CSP(G) is either tractable (if bipartite) or $\mathbb{NP}$-complete.

Theorem (Dichotomy Theorem (Bulatov, Zhuk 17'))

CSP($\mathcal{A}$) *is either tractable or is* $\mathbb{NP}$*-complete.*

Polymorphisms: a motivation to haunt Scott

Question: How does the structure $\mathcal{A}$ effect the complexity of CSP($\mathcal{A}$)?

1. Model theoretic: the ability to "construct" (via certain model theoretic voodoo) K_3 implies $\mathbb{NP}$-hard. Else, it was correctly conjectured to be tractable.

2. Algebraic: The algebraic counterpart is **polymorphisms** of our structure (in the same sense that the algebraic counterpart to definablity is automorphisms). 3. Topological: outside this talk

Polymorphisms

Definition

An n-ary operation $f : A^n \to A$ **preserves** an m-ary relation $\rho \subseteq A^m$ if

$$\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1m} \in \rho \\ a_{21} & a_{22} & \cdots & a_{2m} \in \rho \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{12} & \cdots & a_{1m} \in \rho \\ \downarrow_f & \downarrow_f & \cdots & \downarrow_f \\ X & X & \cdots & X \in \rho \end{array}$$

Definition

Let $(A; \Gamma)$ be a relational structure. An n-ary operation $f : A^n \to A$ is called a **polymorphism** of A if it preserves every relation in Γ.
That is, if f is a homomorphism from A^n to A.
The set of all polymorphisms is denoted $\mathrm{Pol}(A)$.

A simple example

Example

Consider $(A; \neq)$ and $f : A^n \to A$. Then $f \in \mathrm{Pol}(A)$ if and only if

$$x_1 \neq y_1, \ldots, x_n \neq y_n \Rightarrow f(x_1, \ldots, x_n) \neq f(y_1, \ldots, y_n)$$

or, equivalently, if

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \Rightarrow x_i = y_i \text{ for some } 1 \leq i \leq n.$$

We call such a function **injective in one component**.

A Galois connection

Let F be a set of operations on a set A. We denote $\mathbf{Inv(}F\mathbf{)}$ to be the set of relations on A that are invariant under each operation of F.

Lemma

Let $\mathcal{A} = (A; \Gamma)$ and $\mathcal{B} = (A; \Omega)$ be relational structures with the same domain A. Then:

1. *if $Pol(\mathcal{A}) \subseteq Pol(\mathcal{B})$ then $CSP(\mathcal{B})$ is at most as hard as $CSP(\mathcal{A})$.*
2. *$CSP(\mathcal{A})$ and $CSP(Inv(Pol(\mathcal{A})))$ are equally hard.*

Preexisting CSP-semigroup theory pairings

Given an algebraic structure $\mathcal{S} = (S, F)$ we define
$CSP(\mathcal{S}) = CSP(Inv(F))$.
Given the previous lemma, we may only study CSPs of this form.

Theorem (Bulatov, Jeavons, Volkov, 02')

Let S be a finite semigroup. Then $CSP(S)$ is tractable if and only if S is a **block group***, that is, if it does not contain a 2 element left or right zero subsemigroup.*

A few other papers on semigroups:

1. "Tractable clones of polynomials over semigroups" by Dalmau, Gavaldà, Tesson, and Thérien (2005).
2. "Systems of Equations over Finite Semigroups..." by Klìma, Larose and Tesson (2006).

Polymorphisms give tractability

Question: Given a class of structures, when exactly do we have tractability?
Better question: Is the existence of certain types of polymorphisms neccessary for tractability?

Definition

Let f be a 6-ary polymorphism on a relational structure A such that

$$f(x, y, x, z, y, z) = f(y, x, z, x, z, y),$$

then f is called a Siggers term.

Siggers showed (2010) that the lack of a Siggers polymorphism implies $\mathbb{NP}$-completeness.

Theorem (Bulatov, Zhuk 17')

CSP(A) is tractable if and only if Pol(A) contains a Siggers term. Otherwise, CSP(A) is $\mathbb{NP}$-complete.

The countably infinite case

Infinite domains

Many interesting CSPs cannot be formulated by a finite domain.

Example

Consider the acyclic digraph problem. That is, given a finite digraph, is it acyclic? This is equivalent to CSP($\mathbb{Q}; <$), but cannot be written as a CSP with a finite template. Note: the problem is tractable.

Infinite domains allow methods not possible in the finite case:

Example

CSP($\mathbb{N}; \neq$) is tractable (a far cry from the finite case!). The proof transfers an instance into a graph, and uses graph reachability, which is doable in polynomial time.

Problems

Many of the results from the finite case do not transfer to the infinite case:

- There exists problems which are undecidable.
- If a pair of finite relational structures are homomorphically equivalent, then their CSP's are equivalent - not true for infinite structures.
- pp-definablity, extentions by singletons, existance of certain polymorphisms....

A better template: ω-categorical

The structures $(\mathbb{Q}; <)$ and $(\mathbb{N}; \neq)$ have many nice model theoretic properties, including ω-categoricity.

Definition

A structure A is called ω-categorical if it can be uniquely defined, up to isomorphism, by its first order properties. Equivalently: if $\mathrm{Aut}(A)$ is oligomorphic.

Example

The infinite left zero semigroup L is ω-categorical, and is defined by the property $(\forall x)(\forall y)\, xy = x$ (and sentences saying L is infinite). Rectangular bands and null semigroups are also ω-categorical.

By considering ω-categorical structures, we can get back many of the other links from the finite case, including homomorphic equivalence implying equivalent CSPs (Bodirsky, 08').

Pseudo-Siggers

The non-existance of a Siggers term no longer implies $\mathbb{NP}$-completeness (e.g. $(\mathbb{N}; \neq)$).

Definition

A 6-ary operation $f \in$Pol(A) is called a **pseudo-Siggers** term if

$$\alpha f(x, y, x, z, y, z) = \beta f(y, x, z, x, z, y)$$

for some endomorphisms α, β of A.

Lemma (Barto, Pinsker, 16')

If A is ω-categorical and Pol(A) does not contain a pseudo-Siggers term, then CSP(A) is $\mathbb{NP}$-hard.

However, the existance of a pseudo-Siggers term has been shown to be insufficient for tractability. Conjectured true if more conditions are added (reduct of a finitely bounded homogeneous structure).

The big conjecture

Conjecture (Dichotomy Conjecture for ω-categorical strucutres)

The class of ω-categorical structures has CSP dichotomy. That is, CSP(A) is either tractable or $\mathbb{NP}$-hard.

Building examples

Let $(S, \cdot_S)$ be an ω-categorical semigroup. Then we have a few possible CSPs:

1. CSP(Inv($\cdot_S$)). Problem: not ω-categorical.
2. CSP(S;R) where $R = \{(x, y, z) : xy = z\}$. Problem: trivial.
3. CSP(S; R, $\neg R$). Benifits: non-trivial, ω-categorical + other pleasing model theoretic properties- hence known methods for proving complexity in some cases.

We denote the relational structure $(S; R, \neg R)$ by $\bar{S}$.

Lemma (TQG)

Let $f \in Pol(\bar{S})$ of arity n. Then f is a semigroup morphism from S^n to S such that

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \Rightarrow x_i = y_i \text{ for some } 1 \leq i \leq n,$$

where $x_k \in SS$ and $y_k \in S$ $(1 \leq k \leq n)$. Hence if S is regular, then f is injective in one component.

Building examples

Corollary

Let S be a finite semigroup. Then $CSP(\bar{S})$ is tractable if and only if S is either trivial, $|S| = 2$ and non-semilattice, or S is null.

Proof.

Let f be a Siggers term. Then for any $x, y, z \in SS$,

$$f(x, y, x, z, y, z) = f(y, x, z, x, z, y)$$

forcing either $x = y, x = z$ or $y = z$ by our previous lemma. Hence $|SS| \leq 2$. □

Building examples

Example

If L is left zero, then an instance of $CSP(\bar{L})$ is built from $xy = z$ and $xy \neq z$, and thus from $x = z$ and $x \neq z$.

Hence $CSP(\bar{L})$ is equivalent to $\mathrm{CSP}(|L|, \neq)$, and is thus tractable only when $|L| = 1, 2$, or $\aleph_0$.

1. If G is an ω-categorical group, when is $\mathrm{CSP}(G)$ tractable? (motivating example). Known when we have pseudo-Siggers terms.
2. If S is ω-categorical and $\mathrm{CSP}(\bar{S})$ is tractable, is S bi-embeddable with a homogeneous semigroup? (that is, there exists a homogeneous semigroup T and embeddings $\theta : S \to T$ and $\psi : T \to S$.
3. Does $\mathrm{CSP}(S)$ fit with the ω-categorical Dichotomy conjecture?

A wild conjecture

Let L_n (R_n) denote the left (right) zero semigroup with n elements.

Conjecture (TQG)

Let $S = L_n \times R_m$ be a rectangular band for some $n, m \in \mathbb{N} \cup \{\omega\}$. Then $CSP(\bar{S})$ is tractable if and only if S is equal to either

- *Finite: trivial, L_2 or R_2,*
- *$L_n \times R_\omega$ for $n = 1, 2$ or ω,*
- *$L_\omega \times R_m$ for $m = 1, 2$.*

Otherwise, $\bar{S}$ is $\mathbb{NP}$-hard.

In fact this conjecture most likely gives all tractable bands with finite structure semilattice.